3. Let f be analytic in $\bar{B}(0;\ R)$ with $f(0)=0$, $f'(0)\neq 0$ and $f(z)\neq 0$ for $0<|z|\leq R$. Put $\rho=\min\{|f(z)|:|z|=R\}>0$. Define $g:B(0;\ \rho)\to\mathbb{C}$ by

$$g(\omega)=\frac{1}{2\pi i}\int_{\gamma}\frac{zf'(z)}{f(z)-\omega}$$

where γ is the circle $|z|=R$. Show that g is analytic and discuss the properties of g.

4. Prove Caratheodory's Inequality whose statement is as follows: Let f be analytic on $\bar{B}(0; R)$ and let $M(r) = \max\{|f(z)|: |z| = r\}$, $A(r) = \max\{\text{Re} f(z): |z| = r\}$; then for $0 < r < R$, if $A(r) \geq 0$,

$$M(r) \leq \frac{R+r}{R-r}[A(R) + |f(0)|]$$

(Hint: First consider the case where $f(0) = 0$ and examine the function $g(z) = f(Rz)[2A(R) + f(Rz)]^{-1}$ for $|z| < 1$.)

7. Let f be analytic in ann $(0; R_1, R_2)$ and define

$$I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Show that $\log I_2(r)$ is a convex function of $\log r$, $R_1 < r < R_2$.

1. Can you map the open unit disk conformally onto $\{z \mid 0 < |z| < 1\}$?

2. Use theorem 7.2 to give another poof of the Fundamental theorem of algebra

of the a_k may be repeated according to the multiplicity of the zero.) So we can write $f(z) = (z-a_1)(z-a_2)\ldots(z-a_m)g(z)$ where g is analytic on G and $g(z) \neq 0$ for any z in G. Applying the formula for differentiating a product gives

7.1
$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

for $z \neq a_1, \ldots, a_m$. Now that this is done, the proof of the following theorem is straightforward.

7.2 Theorem. *Let G be a region and let f be an analytic function on G with zeros $a_1, \ldots, a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma; a_k)$$

Proof. If $g(z) \neq 0$ for any z in G then $g'(z)/g(z)$ is analytic in G; since $\gamma \approx 0$, Cauchy's Theorem gives $\displaystyle\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$. So, using (7.1) and the definition of the index, the proof of the theorem is finished. □

7.3 Corollary. *Let f, G, and γ be as in the preceding theorem except that* ... $f(z) = \alpha$; *then*